JOURNAL OF COMPUTATIONAL PHYSICS 100, 105–115 (1992)

Extrapolated Surface Charge Method for Capacity Calculation of Polygons and Polyhedra

E. GOTO,*,†,‡ Y. SHI,* AND N. YOSHIDA*,†

* *Department of Information Science, Faculty of Science, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo, 113 Japan;*
† *The Institute of Physical and Chemical Research, 2-1 Hirosawa, Wako-shi, Saitama, 351-01 Japan;*
‡ *Quantum Magneto-Flux Logic Project, Research Development Corporation of Japan, 2-1-42 Ikenohata, Taito-ku, Tokyo, 110 Japan*

Received August 7, 1989; revised July 20, 1990

Effectiveness of extrapolation in calculating the electric capacities of polygons and polyhedra by SCM (surface charge method) is represented. In the case of a square, it is divided into n^2 small squares as treated by Maxwell ($n=6$). Empirically, extrapolation function of the form $\alpha_1/n+\alpha_2/n^2+\beta_1(\log n)/n+\beta_2(\log n)/n^2$ is found to give the best result with the accuracy of more than six decimal figures at $n=28$. In conventional methods without extrapolation, forbiddingly large $n=10^5$ should be needed to obtain the same accuracy. Extrapolation without logarithmic terms ($\beta_1=\beta_2=0$) does not work well. Thus, extrapolation using a logarithmic series and successive refinement leads to both accurate solutions and a saving in computational time. The origin of logarithmic terms is studied. The result of a numerical experiment suggests that logarithmic terms are needed when there are sharp edges in the configuration.

1. INTRODUCTION

The surface charge method (SCM) has been applied to the calculation of electric capacities since Maxwell calculated the capacity of a square by dividing it into 36 small squares [1]. In SCM, the charge distribution on the surface of a conductor is calculated by dividing the surface into many small areas. Although SCM seems to be the most practical method to calculate capacities in the cases where analytical solutions are not known, a vast amount of numerical calculation is needed to obtain accurate values. Furthermore, it is usually difficult to estimate the magnitude of error. Much work has been done to find effective methods to obtain accurate values [2–4].

Recently, Hosoya *et al.* [5] developed an extrapolation technique to calculate inductances in three-dimensional circuit configurations. In their extrapolated surface current method, they divided the surface of a conductor into n small areas in which looping currents flow. In order to obtain an accurate inductance in the limit of $n \to \infty$, they resorted to extrapolation technique using the values calculated with finite values of n.

In this paper, we apply extrapolation technique to SCM calculations of electric capacities of conductors. In Section 2, the formulation of the method is given. As an example, the capacity of a square is treated. Capacities of some polygons and a cube calculated in the same method are presented in Section 3. Problems of extrapolation functions are discussed in Section 4. In particular, two-dimensional cases are studied in order to see the relationship between the logarithmic terms in extrapolation functions and the existence of sharp edges in the system. Concluding remarks are given in Section 5.

2. EXTRAPOLATED SURFACE CHARGE METHOD

The calculation by the extrapolated SCM consists of two steps: SCM and extrapolation. The first step is the calculation of the electric capacity by dividing the surface into many small parts. The second step is the derivation of the true capacity by extrapolating from the capacities calculated by SCM.

2.1. *The Surface Charge Method (SCM)*

The first step is similar to the method taken by Maxwell [1]. In the case of a square, we divide a square into small equal squares. Certain charge distribution is assumed on each small square. Simultaneous linear equations are solved with respect to the charges on small squares so as to make the potential at the middle of each square equal to unity.

In our actual calculation, we divided the square into $(2n)^2$ squares as shown in Fig. 1. From symmetry consideration, we need to know the charges of only one of the four equivalent quarters of a square. We solved the equations for the charges q_{kl} on the square of the kth row and lth line ($1 \leqslant k, l \leqslant n$)

$$\sum_{k=1}^{n} \sum_{l=1}^{n} p_{ij;kl} q_{kl} = 1 \qquad \text{for} \quad 1 \leqslant i, j \leqslant n, \tag{1}$$

0021-9991/92 $5.00

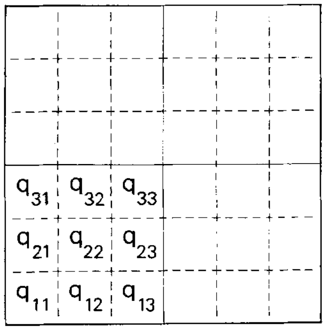


FIG. 1. Division of a square ($2n = 6$).

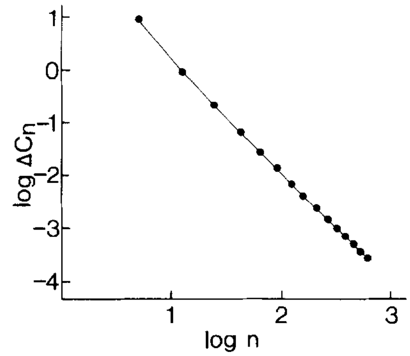


FIG. 3. The values of log ΔC_n plotted as a function of log n.

where $p_{ij;kl}$ is the potential at the middle of the (ij)th square due to the unit charge on the (kl)th square. We assumed uniform charge distribution on each small square. An explicit form of $p_{ij;kl}$ is given in Appendix A.1, together with its derivation. The total charge is the approximate value of the capacity C_n of the original square. Namely,

$$C_n = \sum_{i,j} q_{ij} \tag{2}$$

which, in the present case, reduces to

$$C_n = 4 \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij}. \tag{3}$$

The approximate capacities C_n thus calculated for $1 \leqslant n \leqslant 16$ are plotted in Fig. 2.

2.2. *The Extrapolation*

In the second step of the extrapolated SCM, we derive the true capacity $C = \lim_{n \to \infty} C_n$ from the calculated C_n's. The extrapolation technique is essentially similar to that

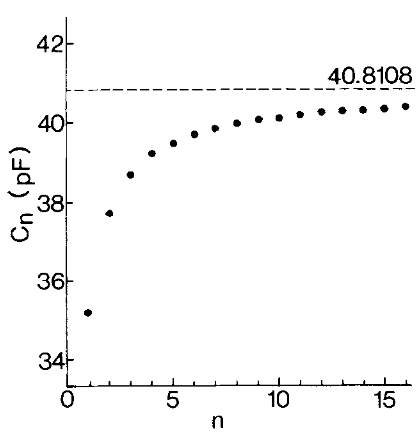


FIG. 2. Electric capacities C_n of a square with its sides of 1 m calculated by the SCM by dividing it into $(2n)^2$ small equal squares. The dashed line shows the value obtained by extrapolation ($n \to \infty$).

taken by Hosoya *et al.* [5]. Namely, we calculate $C_\infty = \lim_{n \to \infty} C(n)$ by assuming an extrapolation function $C(n)$ of n and requiring that $C(n) = C_n$ at calculated points of n.

In order to look at the gross behavior of C_n, we plot $\log |\Delta C_n| \approx \log |dC(n)/dn|$ as a function of $\log n$ in Fig. 3, where the difference operation Δ is defined as $\Delta f(n) = f(n) - f(n-1)$ for any function $f(n)$ of n. As seen in the figure, the points are almost connected by a straight line, which suggests that $dC(n)/dn$ is approximated by some power of n. Since the gradient is about -2, we see that the leading n-dependent part of $C(n)$ should be proportional to $1/n$, and we expect that it is improved by including higher-order in $1/n$.

Taking a polynomial in $1/n$,

$$C(n) = C_\infty + \sum_{i=1}^{\infty} \alpha_i \frac{1}{n^i}, \tag{4}$$

we calculated C_∞ from the $C_n, C_{n-1}, ..., C_{n-k}$ by substituting them for $C(n), C(n-1), ..., C(n-k)$:

$$C_\infty \approx C_\infty^{(k)}(n) \equiv \frac{1}{k!} \Delta^k (n^k C_n). \tag{5}$$

The derivation of Eq. (5) and some related formulas are given in Appendix B.1. If we take $k = 3$ and $n = 16$, for example, and use $C_{16} = 40.38265752524515$ pF, $C_{15} = 40.35484913286501$ pF, $C_{14} = 40.32315569495833$ pF, and $C_{13} = 40.28669931627749$ pF, we have $C_\infty = 40.80951$ pF. Other parameters in Eq. (4) are $\alpha_1 = -6.9914$ pF, $\alpha_2 = 2.828$ pF, $\alpha_3 = -3.83$ pF, which indicate that the n-dependent part is actually dominated by α_1/n.

If C_n is completely represented by a polynomial of kth degree in $1/n$, $C_\infty^{(k)}(n)$ should be independent of n. The plotted curves, however, show some deviation from constant. In order to magnify the deviation, in Fig. 4 we plot the values of $\log \Delta^{k+1}(n^k C_n) \approx \log[k!\, \Delta C_\infty^{(k)}(n)]$ as a

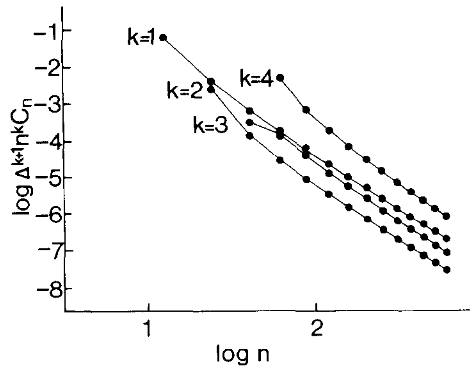


FIG. 4. The values of $\log(\Delta^{k+1}(n^k C_n))$ (Eq. (5)) plotted as a function of $\log n$, for $k = 1, 2, 3$, and 4.

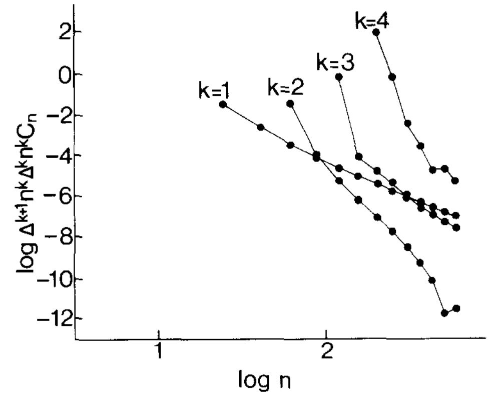


FIG. 5. The values of $\log(\Delta^{k+1}(n^k\Delta^k(n^k C_n)))$ (Eq. (9)) plotted as a function of $\log n$ for $k = 1, 2, 3$, and 4.

function of $\log n$, for $k = 1, 2, 3$, and 4. As seen in the figure, the points for each k are connected by an almost straight line. This fact implies that the error is well approximated by a term proportional to a certain power of $1/n$. Assuming that the error is given by γ/n^x where γ and x are constants, we have

$$\Delta^{k+1}(n^k C_n) \approx (-1)^{k+1} \frac{\Gamma(x+1)}{\Gamma(x-k)} \frac{\gamma}{n^{x+1}}. \qquad (6)$$

Therefore, the gradient of $\log |\Delta^{k+1}(n^k C_n)|$ is $-(x+1)$. The magnitude of error in $C_\infty^{(k)}(n)$ by Eq. (5) can be estimated to be

$$\varepsilon \equiv |C - C_\infty^{(k)}(n)| \approx \frac{1}{k!} \left| \frac{n}{x} \Delta^{k+1}(n^k C_n) \right|, \qquad (7)$$

where C is the exact capacity.

When k is increased, however, the gradient of the curve stays around -3 even when k is increased, i.e., the convergence is not improved by taking higher-order terms in Eq. (4). As explained in Appendix B.3, this suggests the lack of logarithmic terms in the assumed extrapolation function (may be called "the logarithmic syndrome"). We therefore try an extrapolation function with logarithmic terms. In this case, the extrapolation function can be written as

$$C(n) = C_\infty + \sum_{i=1}^{\infty} \alpha_i \frac{1}{n^i} + \sum_{i=1}^{\infty} \beta_i \frac{\log n}{n^i}. \qquad (8)$$

The approximate value of the extrapolated capacity C_∞ determined from $C_n, C_{n-1}, \ldots, C_{n-2k}$ is (see Appendix B.2)

$$C_\infty \approx C_\infty^{(k)}(n) \equiv \frac{1}{(k!)^2} \Delta^k(n^k \Delta^k(n^k C_n)). \qquad (9)$$

To estimate the magnitude of errors, we plot $\log(\Delta^{k+1}(n^k\Delta^k(n^k C_n))) \approx \log[(k!)^2 \Delta C_\infty^{(k)}(n)]$ against $\log n$ in Fig. 5, for $k = 1, 2, 3$, and 4. Assuming the error term of the form γ/n^x, we have

$$\Delta^{k+1}(n^k\Delta^k(n^k C_n)) = -\frac{\Gamma(x)\,\Gamma(x+1)}{\Gamma(x-k)^2} \frac{\gamma}{n^{x+1}}. \qquad (10)$$

Thus we can estimate the error by

$$\varepsilon \approx \frac{1}{(k!)^2} \left| \frac{n}{x} \Delta^{k+1}(n^k\Delta^k(n^k C_n)) \right|. \qquad (11)$$

The irregular behaviour seen at $n \geqslant 15$ in Fig. 5 can be attributed to the round-off error in numerical computation. Irregularity is also seen at the smaller side of n. This may reflect some effects of discrete approximation. We throw away these irregular parts and take smooth parts only. As seen in this figure, the gradient is steepest in the curve with $k = 4$. The curve with $k = 3$ is less steep than that with $k = 2$. Although the part of $n \leqslant 14$ of the curve with $k = 4$ looks all right, the behaviour around $n = 13$ is somewhat irregular. To be on the safe side, we adopt the point with $n = 14$ (division into 28×28 squares) on the curve for $k = 2$. In this case, the gradient of $\log [\Delta C^{(k)}(n)|$ is around -8, so that the error term is proportional to $1/n^7$ and the error is estimated to be $\varepsilon \approx 2 \times 10^{-5}$ pF. The corresponding value of the extrapolated capacity C_∞ is 40.81083 pF by Eq. (9), with an error estimate of 5×10^{-5}%.

The capacity obtained in the present work is comparable to 40.48 pF by Maxwell [1] and 40.58 pF by Ruehli *et al.* [2]. If extrapolation were not applied, the accuracy of $\varepsilon \approx 2 \times 10^{-5}$ pF would be attained only by the division of as large as $n \approx |\alpha_1/\varepsilon| \approx 3.5 \times 10^5$.

We have also tried extrapolation functions with terms like $1/n^{i/2}$ though we have not seen remarkable improvement.

2.3. *The Extrapolated SCM*

In the following, we summarize the procedure to calculate capacities by the extrapolated SCM.

(i) Devise a scheme of dividing the surface of the conductor into many squares or triangles. Introduce a parameter n so that the length of the edges of each small square or triangle is proportional to $(1/n)$ and so that the way of division is definitely determined when n is given.

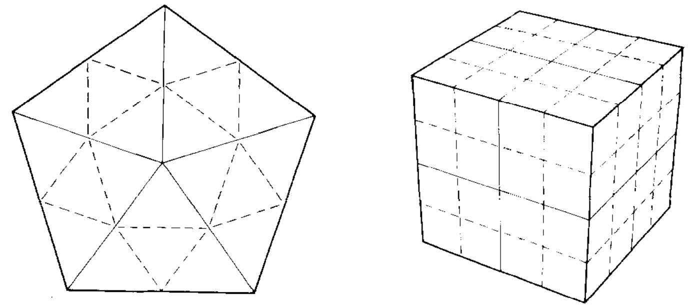
(ii) Calculate the capacities C_n by the SCM for $n = 1, 2, \ldots,$ up to a reasonably large value of n.

(iii) Plot $\log |\Delta^{k+1}(n^k C_n)|$ versus $\log n$ for $k = 1$, then $k = 2$, and so on. Take the smooth part of the plot. If the points show irregular behavior, consider it as originating from the round-off error, or as n being too small, and throw away these parts. If the gradient gets steeper with increasing k, then adopt as the capacity the values of $C_\infty^{(k)}(n)$ by Eq. (5) with k and n by which the error estimate is smallest. The errors can be estimated from the plot of $\log |\Delta^{k+1}(n^k C_n)|$ using Eq. (7).

(iv) If the gradient of $\log |\Delta^{k+1}(n^k C_n)|$ does not get steeper even when k is increased ("the logarithmic syndrome"), then plot $\log |\Delta^{k+1}(n^k \Delta^k(n^k C_n))|$ versus $\log n$ for successively larger k as long as the curve does not show an irregular behavior. Adopt as the capacity the value of $C_\infty^{(k)}(n)$ calculated by Eq. (9) with k and n by which the error estimate is smallest. The errors can be estimated from the plot of $\log |\Delta^{k+1}(n^k \Delta^k(n^k C_n))|$ using Eq. (11).

So far, we have treated only conductors which are constructed by flat surfaces and straight lines only. The application of the method to those conductors which have smoothly curved surfaces can be made by taking some points on the surface and approximating the curved surfaces by sets of flat surfaces determined by these points. If the system has multiple scale length, first divide the conductors into a number of rectangles and/or triangles, each of which has a consistent scale. Then, sub-divide each part into n^2 small sections.

The extrapolated SCM can also be applied to the systems with multiple conductors. Namely, to calculate capacity coefficients $C_{\beta\alpha}$ $(1 \leqslant \beta \leqslant N)$ of the αth conductor $(1 \leqslant \alpha \leqslant N)$ of an N-conductor system, divide the surface of each conductor into $n \times n$ squares (or triangles). Then solve the following set of equations for the charges $q_{kl(\beta)}$: $\sum p_{ij(\alpha');kl(\beta)} q_{kl(\beta)} = \delta_{\alpha'\alpha}$ for $1 \leqslant \alpha' \leqslant N$, where $p_{ij(\alpha');kl(\beta)}$ is the potential at the middle of the (ij)th square on the surface of the α'th conductor due to unit charge at the (kl)th square on the surface of the βth conductor. The capacity coefficient can be calculated by $C_{\beta\alpha} = \sum_{ij} q_{ij(\beta)}$. In some multi-conductor systems, the potential is defined relative to one of the conductors that surrounds all the others. Some examples of such a case are treated in Section 4.



(a) (b)

FIG. 6. Division of (a) a pentagon and (b) a cube. The fine full lines show the initial division of the surface, while the dashed lines show the sub-division for $n = 2$.

3. EXAMPLES OF APPLICATION

Capacities of other polygons can be calculated by dividing them into equal triangles. In this way, we have calculated the capacities of an equilateral triangle, a square, an equilateral pentagon, an equilateral hexagon, an equilateral heptagon, and an equilateral octagon up to seven decimal digits by dividing them into small triangles. The division of a pentagon is shown in Fig. 6a as an example. The results are summarized in Table I. Note that the capacity is approaching that of a unit circle for which the exact value is known to be $C = 8\varepsilon_0 \approx 70.83350$ pF.

We also calculated the capacity of a cube by dividing each surface into $(2n)^2$ equal squares as shown in Fig. 6b. The result is included in Table I. The obtained value is close to the lower bound 72.9 pF set by Reitan *et al.* [4]. The application to other polygons and polyhedra are straightforward.

Table I may be used for benchmark tests of various approximation methods. We hope that the capacities tabulated here will be used to test the accuracy of various approximation techniques such as those used in the finite element method.

TABLE I

The Capacities of Equilateral Polygons Which Internally Touch a Circle with a Radius of 1 m, and the Capacity of a Cube with Edge of 1 m Calculated by the Extrapolated SCM

Shape	C_∞ (pF)	Relative error
Triangle	48.34699	1×10^{-6}
Square	57.71519	3×10^{-6}
Pentagon	62.24622	2×10^{-6}
Hexagon	64.78146	8×10^{-7}
Heptagon	66.34090	2×10^{-7}
Octagon	67.36759	1×10^{-7}
Cube	73.50997	8×10^{-7}

4. DISCUSSION

Based on a conjecture that the logarithmic terms in extrapolation functions may be related to singularities in charge distribution at sharp edges, we made a numerical experiment on conductors with and without sharp edges.

For simplicity, we tried it in two dimensions. The example is the capacity of an infinitely-long right m-prism located parallel to an infinitely-long right circular cylinder which has the same central axis with the prism. Figure 7 explains the cross section of that configuration. The radius of the cylinder is r_2, while the prism has the size to inscribe a circular cylinder of a radius r_1. The system is infinitely long in the direction perpendicular to this plane. Figures 8 and 9 show two cases of this system ($m = 6$). In the configuration of Fig. 8, a hexagonal prism is placed in a circular cylinder. In this case, the prism has sharp edges at the outer corners. In Fig. 9, on the other hand, a hexagonal prism surrounds a circular cylinder, where the edges are at the inner corners so that the angles are larger than π. In these two cases, the conductors have sharp edges. When $m \to \infty$, these m-prisms turn out to be circular cylinders, as shown in Fig. 10. This is an example which does not have any sharp edges.

In two-dimensional cases, the potential is defined relative to the other conductor. Let q_{ij} be the charge per unit length on the m-prism ($i = 1$) or on the cylinder ($i = 2$). Equations (1) are modified to the following $2n + 1$ equations for q_{ij} ($1 \leqslant i \leqslant 2$, $1 \leqslant j \leqslant n$) and the potential P. We solve

$$\begin{aligned} &\sum_{k=1}^{2} \sum_{l=1}^{n} p_{1i;kl} q_{kl} = P + 1 \qquad \text{for} \quad 1 \leqslant i \leqslant n, \\ &\sum_{k=1}^{2} \sum_{l=1}^{n} p_{2i;kl} q_{kl} = P \qquad \text{for} \quad 1 \leqslant i \leqslant n, \qquad (12) \\ &\sum_{k=1}^{2} \sum_{l=1}^{n} q_{kl} = 0, \end{aligned}$$

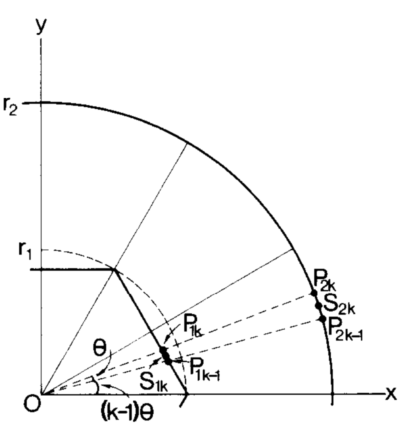


FIG. 7. The definition of the points used in the calculation of the capacities of the systems shown in Figs. 8, 9, and 10. The points S_{ij} are the source points, while the points P_{ij} are the field points. The present figure corresponds to the case of Fig. 8.

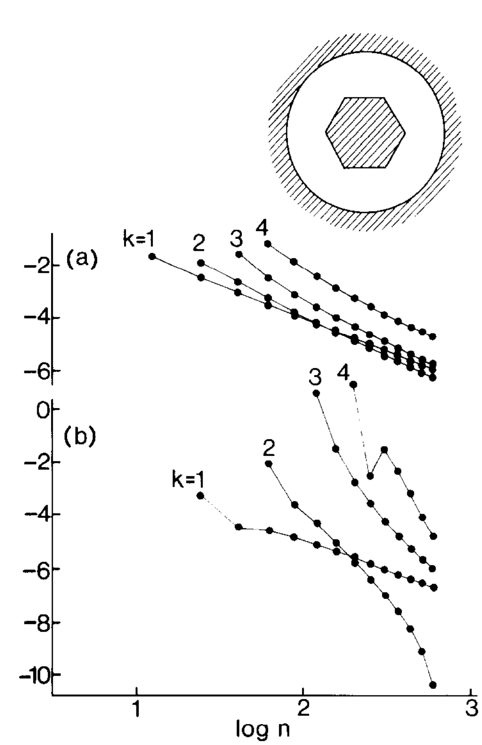


FIG. 8. The figure shows (a) $\log(\Delta^{k+1}(n^k C_n))$ and (b) $\log(\Delta^{k+1}(n^k \Delta^k(n^k C_n)))$ for $k = 1, 2, 3$, and 4 plotted as a function of $\log n$. The values of C_n are calculated by SCM for the system where a hexagonal prism ($r_1 = 1$) is surrounded by a circular cylinder ($r_2 = 2$) as shown in the upper-right part of the figure.

where $p_{ij;kl}$ stands for the potential at the field point (ij) due to the unit charge density of line charge at the source point (kl). The explicit forms of $p_{ij;kl}$ are given in Appendix C. The electric capacity per unit length is obtained by summing up the charges on the prism:

$$C = 2m \sum_{i=1}^{n} q_{1n}. \qquad (13)$$

Figure 8 shows the case where a hexagonal prism ($r_1 = 1$ m) is in a circular cylinder ($r_2 = 2$ m). The values of $\log |\Delta^{k+1}(n^k C_n)|$ are plotted in (a) for $k = 1, 2, 3$, and 4. The gradients for $k = 2, 3$, and 4 almost stay around -3, indicating "the logarithmic syndrome." The quantities $\log |\Delta^{k+1}(n^k \Delta^k(n^k C_n))|$ are plotted in (b). In this case, on the other hand, the gradient is generally increasing from around -2 for $k = 1$ to around -12 for $k = 4$, except for some irregularity around $n = 10$. The estimated error at $n = 16$ for $k = 4$ by polynomial extrapolation function is about 130 times larger than that by the extrapolation function with logarithmic terms. This fact also supports the necessity of logarithmic terms.

Figure 9 shows the situation where a cylinder ($r_2 = 1$ m) is surrounded by a hexagonal prism ($r_1 = 2$ m). This situation is different from the previous one in the point that this

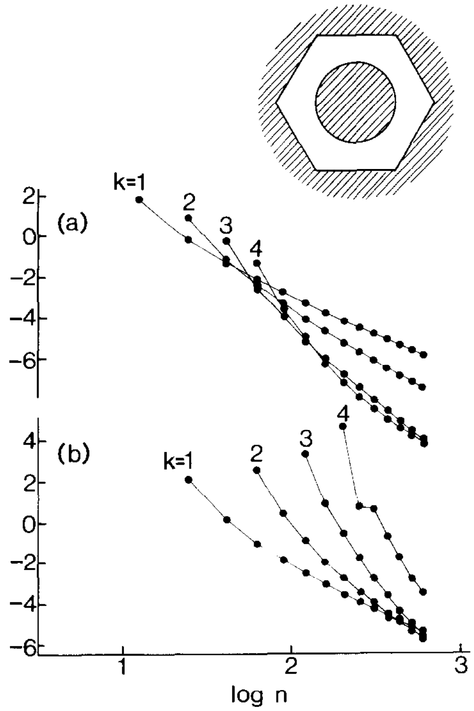


FIG. 9. The figure shows (a) $\log(\Delta^{k+1}(n^k C_n))$ and (b) $\log(\Delta^{k+1}(n^k \Delta^k(n^k C_n)))$ for $k = 1, 2, 3$, and 4 plotted as a function of $\log n$. The values of C_n are calculated by SCM for the system where a circular cylinder ($r_2 = 1$) is surrounded by a hexagonal prism ($r_1 = 2$) as shown in the upper-right part of the figure.

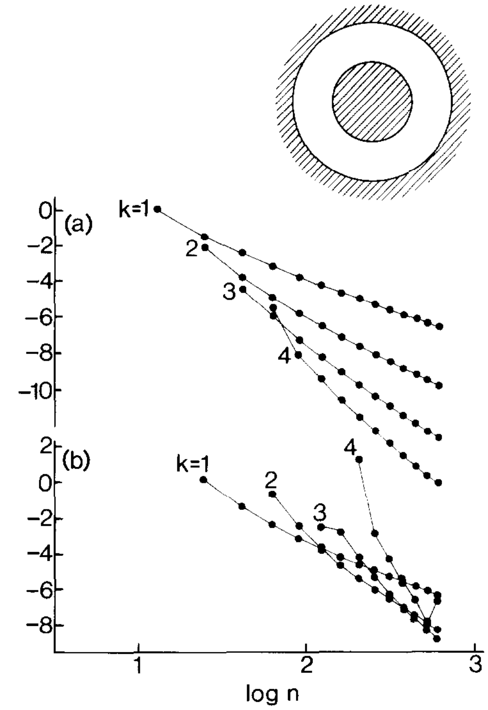


FIG. 10. The figure shows (a) $\log(\Delta^{k+1}(n^k C_n))$ and (b) $\log(\Delta^{k+1}(n^k \Delta^k(n^k C_n)))$ for $k = 1, 2, 3$, and 4 plotted as a function of $\log n$. The values of C_n are calculated by SCM for the system where a circular cylinder ($r_1 = 1$) is surrounded by another circular cylinder ($r_2 = 2$) as shown in the upper-right part of the figure.

conductor has edges at the outer corners while the previous conductors had edges at the inner corners. The result of the calculation is, however, similar to the previous one. By the polynomial extrapolation functions (Fig. 9a), the gradient of $\log |\Delta C_\infty^{(k)}(n)|$ does not become steep enough, while by the extrapolation functions with logarithmic terms (Fig. 9b) the gradient of $\log |\Delta C_\infty^{(k)}(n)|$ becomes steep enough with increasing k. This means that logarithmic terms are needed in the extrapolation function.

For comparison with the above two cases with edges, we calculated the capacity between two circular cylinders. The smaller one has a radius of 1 m, while the large one has a radius of 2 m. In this case, there is no sharp edge. Dividing the surface of each cylinder into $12n$ equivalent parts, we calculated the capacity assuming a line charge of q and $-q$ per unit length in the middle of each arc and determined q so as to make the potential difference between the two cylinders equal to unity. The values of $\log |\Delta^{k+1}(n^k C_n)|$ are plotted in Fig. 10a for $k = 1, 2, 3$, and 4. It is seen that the gradient becomes steeper from about -3.5 ($k = 1$) to about -8.5 ($k = 4$). This tendency implies that the polynomial form is a reasonable approximation to $C(n)$. Incidentally, the extrapolated value at $n = 16$ for $k = 4$ ($C/2\pi\varepsilon_0 = 1.4426950413 \pm 0.0000000004$) is very close to the exact value: $C/2\pi\varepsilon_0 = 1/\log 2 \approx 1.4426950409$.

Comparing the results, we see that the extrapolation functions without logarithmic terms do not work well when there are edges. This is observed for either outer or inner corners whenever the conductors have sharp edges. On the other hand, polynomial form seems to be valid for extrapolation functions if all parts are smooth. The occurrence of logarithmic terms might be related to the singularities in charge distribution at sharp edges. Further pursuit of this problem, however, will be a theme of future study.

5. CONCLUDING REMARKS

We have presented the extrapolated SCM and obtained the capacity of a square with its sides of 1 m to be 40.81083 ± 0.00002 pF. The extrapolation reduces the computation a great deal. If we divide a square into n^2 small squares, the number of elements of the simultaneous equation, i.e., the size of the matrix, is of the order of $O(n^2)$. Thus the number of matrix elements is $O(n^4)$, and the amount of computation for solving the equation is $O(n^6)$. As we have seen, the magnitude or error is $O(1/n)$ if no extrapolation is made. Thus the order of $O(\varepsilon^{-6})$ amount of computation is needed to make ε small. In the proposed method, on the other hand, the error is $O(n^{-8})$, so that only $O(\varepsilon^{-6/8})$ computation is needed to obtain the same accuracy. More quantitative estimation shows that the present result of the capacity of a square with $n \approx 14$ is almost equivalent to $n \approx 10^5$ without extrapolation. Although errors cannot be easily estimated in many cases by the usual methods, by using extrapolation, error can be estimated easily.

The extrapolation technique for calculating capacities can be applied, in principle, to other methods if they involve a certain parameter "n" by which exact capacity is obtained in the limit of $n \to \infty$. For example, the present method, SCM, belongs to the method of moments by appropriately selecting the set of basic functions. In more general cases, the extrapolation technique can be used, making "n" the number of trial functions. The surface solvers as used in the present work will be more appropriate than volume solvers, since surface division (n^2) has elements less than volume division (n^3). The advantage of using extrapolation is that not much accuracy is required in "direct" calculated results by finite values of n. Therefore, elaboration in determining the grid distribution is not needed. In particular, in many of other works on techniques in this kind of computation, special treatment is made near the edge in order to obtain better accuracy by small numbers of divisions [2, 3]. Since we are deriving the value at $n \to \infty$ by extrapolation, the values with finite divisions need not be very near to the true capacity. Rather, such irregular treatment might give rise to irregular behavior of C_n which we would like to avoid.

A drawback of the present method is that the true extrapolation function is not known. From the results of Section 4, we may consider that the necessity of logarithmic terms is closely related to the existence of sharp edges. This argument is, however, based only on the results of very few cases, and there seems to be no theory to justify it yet. As the extrapolation function, we tried fractional powers of n to some extent, and it seemed that they are not essential. We have not studied, however, terms like $(\log n)^2/n^k$, nor oscillating functions such as $\sin(\alpha n)/n^k$. The form of extrapolation functions is an open problem. It is desired that the study of extrapolation be made in the near future, both numerically and analytically.

To summarize, we have presented the extrapolated surface charge method as a practical method to obtain accurate electric capacities economically. We have observed the effectiveness of extrapolation which enables one to calculate capacities to the accuracy where usual methods without extrapolation cannot reach. The magnitude of error can be estimated easily. One interesting subject for future study is the form of extrapolation functions, especially, the relationship among logarithmic terms, singularities in charge distribution, and sharp edges in the configuration. It is desired that the study of extrapolation functions be made.

Finally, we express our gratitude to Mr. J. Uchikawa and Mr. N. Kurita for their help in computing.

APPENDIX A: POTENTIAL BY UNIFORM CHARGE ON A SURFACE

A.1. *Square*

The potential produced by uniformly distributed charge q on a rectangle made from points (x_1, y_1), (x_1, y_2), (x_2, y_1), and (x_2, y_2) on the x-y plane is given by

$$V = \frac{1}{4\pi\varepsilon_0} \frac{q}{(x_2 - x_1)(y_2 - y_1)} \times I(X, Y, Z; x_1, y_1; x_2, y_2), \tag{A.1}$$

where $4\pi\varepsilon_0 = 10^3/2.99792458^2$ pF/m ≈ 111.265006 pF/m, I is the surface integral,

$$I(X, Y, Z; x_1, y_1; x_2, y_2) = \int \frac{1}{r}\, dS = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \frac{1}{r}\, dx\, dy, \tag{A.2}$$

and r is the distance between the source point $S(x, y, 0)$ and the field point $P(X, Y, Z)$:

$$r = [(X - x)^2 + (Y - y)^2 + Z^2]^{1/2}. \tag{A.3}$$

This integral can be calculated from the indefinite integral,

$$\iint \frac{1}{r}\, dx\, dy = F(X, Y, Z; x, y) = x \log(r + y) + y \log(r + x), \tag{A.4}$$

resulting in

$$\begin{aligned} I(X, Y, Z; x_1, y_1; x_2, y_2) &= F(X, Y, Z; x_2, y_2) - F(X, Y, Z; x_2, y_1) \\ &\quad - F(X, Y, Z; x_1, y_2) + F(X, Y, Z; x_1, y_1). \end{aligned} \tag{A.5}$$

Now, we come to the case of the square of Fig. 1. We have the coefficient $p_{ij;kl}$ in Eq. (1) as

$$\begin{aligned} p_{ij;kl} &= v_{ij;kl} + v_{ij;k,2n+1-l} \\ &\quad + v_{ij;2n+1-k,l} + v_{ij;2n+1-k,2n+1-l} \\ &\quad \text{for} \quad 1 \leqslant i, j, k, l \leqslant n, \end{aligned} \tag{A.6}$$

where

$$v_{ij;kl} = \frac{1}{4\pi\varepsilon_0} \frac{1}{(2n)^2} \times I\left(\frac{i - 1/2}{2n}, \frac{j - 1/2}{2n}, 0; \frac{k-1}{2n}, \frac{l-1}{2n}; \frac{k}{2n}, \frac{l}{2n}\right). \tag{A.7}$$

In Eq. (A.6), the contributions from three squares $(k, 2n+1-l)$, $(2n+1-k, l)$, and $(2n-1-k, 2n+1-l)$ are included, in addition to that from (k, l).

A.2. *Triangle*

The potential produced by the uniformly distributed charge on the triangle ΔABC as shown in Fig. 11 is derived as follows: Although we are presently concerned with the cases where field points and source points are all in the same plane, we consider a more general case where the field point is not necessarily in the plane of the triangle. The potential is given by

$$V = \frac{1}{4\pi\varepsilon_0} \frac{q}{\Delta ABC} I, \tag{A.8}$$

where I is the integral by the source point S over the surface of the triangle,

$$I = \int_{\Delta ABC} \frac{1}{r}\, dS, \tag{A.9}$$

r being the distance between S and the field point P. This integral can be calculated by taking the sum (or the difference) of those over three triangles,

$$I = \pm I_{\Delta ABG} \pm I_{\Delta BCG} \pm I_{\Delta CAG}, \tag{A.10}$$

where G is the foot of P on the plane of ΔABC. Now, we calculate $I_{\Delta QRG}$ for $QR = AB$, BC, or CA. In the following, we shall use Y_X to denote the vector from a point X to a point Y. The coordinate (u, v) defined as

$$S_G = vQ_G + uvR_Q \tag{A.11}$$

is convenient for evaluating the integral over the triangle ΔQRG. The integral becomes

$$\pm I_{\Delta QRG} = \int_0^1 du \int_0^1 v\, dv \frac{\sigma}{[q(u)\, v^2 + h^2]^{1/2}}, \tag{A.12}$$

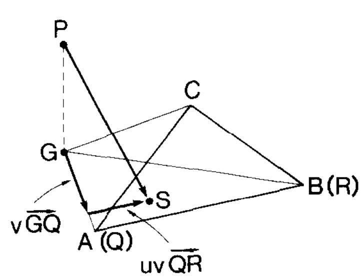


FIG. 11. The calculation of the potential at the point P due to uniform charge on the triangle ABC. The point G shows the foot of the perpendicular of the plane ABC.

where

$$q(u) = |R_Q u + Q_G|^2, \qquad \sigma = (Q_G \times R_G) \cdot n, \tag{A.13}$$

h is the distance between P and G, and n is the unit vector in the direction of the vector $(A_C \times B_C)$. Note that the magnitude of σ is twice the area of the triangle QRG, while its sign is positive (negative) if the direction of the loop GQR is the same as (opposite to) that of loop ABC. After integrating over v, we have

$$I_{\Delta QRG} = I_1 - I_0, \tag{A.14}$$

where

$$I_0 = \sigma h \int_0^1 du \frac{1}{q(u)}, \tag{A.15}$$

and

$$I_1 = \sigma \int_0^1 du \frac{[q(u) + h^2]^{1/2}}{q(u)}. \tag{A.16}$$

The integral I_1 can be separated into two integrals,

$$I_1 = I_2 + I_3, \tag{A.17}$$

where

$$I_2 = \sigma \int_0^1 du \frac{1}{[q(u) + h^2]^{1/2}} \tag{A.18}$$

and

$$I_3 = \sigma h^2 \int_0^1 du \frac{1}{q(u)[q(u) + h^2]^{1/2}}. \tag{A.19}$$

The integration of I_0, I_2, and I_3 can be carried out in an elementary way. As a result, we obtain the following formula:

$$\begin{aligned} I_{\Delta QRG} = {} & \frac{\sigma}{|R_Q|} \log \frac{R_Q \cdot R_G + |R_Q|\,|R_P|}{-Q_R \cdot Q_G + |Q_R|\,|Q_P|} \\ & + h \arctan \frac{\sigma\, R_Q \cdot R_G (h - |R_P|)}{\sigma^2\, |R_P| + h(R_Q \cdot R_G)^2} \\ & + h \arctan \frac{\sigma Q_R \cdot Q_G (h - |Q_P|)}{\sigma^2\, |Q_P| + h(Q_R \cdot Q_G)^2}. \end{aligned} \tag{A.20}$$

If the field point P is on the plane of ΔABC, P coincides with

G. This case corresponds to the limit of $h \to 0$, where Eq. (A.20) reduces to

$$I_{\Delta QRP} = \frac{\sigma}{|R_Q|} \log \frac{R_Q \cdot R_P + |R_Q|\,|R_P|}{-Q_R \cdot P_P + |Q_R|\,|Q_P|}. \tag{A.21}$$

APPENDIX B. DERIVATION OF C_∞

B.1. *Extrapolation by Polynomials*

As in Ref. [5], we take the extrapolation function of polynomial of kth degree of $(1/n)$:

$$C(n) = C_\infty + \sum_{i=1}^{\infty} \alpha_i \frac{1}{n^i}. \tag{B.1}$$

Because

$$\frac{d^k}{dn^k} n^k = \Delta^k n^k = k! \tag{B.2}$$

and

$$\frac{d^k}{dn^k} n^{k-i} = \Delta^k n^{k-i} = 0 \qquad \text{for} \quad 1 \leqslant i \leqslant k, \tag{B.3}$$

the terms in Eq. (B.1) with $i \leqslant k$ vanish by the following operation:

$$\begin{aligned} \frac{1}{k!} \Delta^k(n^k C(n)) &= \frac{1}{k!} \frac{d^k}{dn^k}(n^k C(n)) + O\left(\frac{1}{n^{k+1}}\right) \\ &= C_\infty + O\left(\frac{1}{n^{k+1}}\right). \end{aligned} \tag{B.4}$$

By substituting C_n, C_{n-1}, ..., C_{n-k} for $C(n)$, $C(n-1)$, ..., $C(n-k)$ and neglecting $O(1/n^{k+1})$, we have

$$C_\infty \approx C_\infty^{(k)}(n) \equiv \frac{1}{k!} \Delta^k(n^k C_n). \tag{B.5}$$

In order to estimate the error in $C_\infty^{(k)}(n)$, we introduce the error function $g(n)$:

$$C_n = C + \sum_{i=1}^{k} \frac{\alpha_i}{n^i} + g(n). \tag{B.6}$$

Then, $C_\infty^{(k)}(n)$, calculated by Eq. (B.5), becomes

$$\begin{aligned} C_\infty^{(k)}(n) &= C + \frac{1}{k!} \Delta^k(n^k g(n)) \\ &= C + \frac{1}{k!} \frac{d^k}{dn^k}(n^k g(n)) \\ &\quad + O\left(\frac{d^{k+1}}{dn^{k+1}}(n^k g(n))\right). \end{aligned} \tag{B.7}$$

Thus the leading part of the error in Eq. (B.5) is

$$\begin{aligned} \varepsilon &\equiv |C_\infty^{(k)}(n) - C| \\ &\approx \left| \frac{1}{k!} \frac{d^k}{dn^k}(n^k g(n)) \right|. \end{aligned} \tag{B.8}$$

The true form of $g(n)$ is not known in general. Empirically, however, it can often be approximated by a certain power of $1/n$. If $g(n) = \gamma/n^x$ with $x > k$, then

$$\varepsilon \approx \left| \frac{\gamma}{k!} \frac{\Gamma(x)}{\Gamma(x-k)} \frac{1}{n^x} \right| + O\left(\frac{1}{n^{x+1}}\right) \tag{B.9}$$

and

$$\begin{aligned} \Delta^{k+1}(n^k C_n) &= \Delta^{k+1}(n^k g(n)) \\ &\approx \frac{d^k}{dx^k}(n^k g(n)) \\ &= \gamma(-1)^{k+1} \frac{\Gamma(x+1)}{\Gamma(x-k)} \frac{1}{n^{x+1}}. \end{aligned} \tag{B.10}$$

Thus we have the estimation of error:

$$\varepsilon \approx \frac{1}{k!} \left| \frac{n}{x} \Delta^{k+1}(n^k C_n) \right|. \tag{B.11}$$

For other forms of $g(n)$, see Ref. [5], where the effects of the round-off error are also discussed.

B.2. *Extrapolation with Logarithmic Terms*

We take the following extrapolation function:

$$C(n) = C_\infty + \sum_{i=1}^{\infty} \alpha_i \frac{1}{n^i} + \sum_{i=1}^{\infty} \beta_i \frac{\log n}{n^i}. \tag{B.12}$$

Using the relation between difference and differentiation,

$$\begin{aligned} \Delta^k f(n) &= \frac{d^k}{dn^k} f(n) - \frac{k}{2} \frac{d^{k+1}}{dn^{k+1}} f(n) \\ &\quad + \frac{k(3k+1)}{24} \frac{d^{k+2}}{dn^{k+2}} f(n) + \cdots, \end{aligned} \tag{B.13}$$

we obtain from Eqs. (B.2), (B.3), and

$$\frac{d^k}{dn^k} n^{k-i} \log n = (-1)^{i-1} (k-i)!\,(i-1)! \frac{1}{n^i} \tag{B.14}$$

the following formula:

$$\frac{1}{(k!)^2} \Delta^k(n^k \Delta^k(n^k C(n)))$$
$$= \frac{1}{(k!)^2} \frac{d^k}{dn^k}\left(n^k \frac{d^k}{dn^k}(n^k C(n))\right) + O\left(\frac{1}{n^{k+1}}\right)$$
$$= C_\infty + O\left(\frac{1}{n^{k+1}}\right). \tag{B.15}$$

Substituting C_n, $C_{n-1}, ..., C_{n-2k}$ for $C(n)$, $C(n-1), ...,$ $C(n-2k)$ and neglecting $O(1/n^{k+1})$, we obtain

$$C_\infty \approx C_\infty^{(k)}(n) \equiv \frac{1}{(k!)^2} \Delta^k(n^k \Delta^k(n^k C_n)). \tag{B.16}$$

The error $g(n)$, where

$$C_n = C + \sum_{i=1}^{k} \frac{\alpha_i}{n^i} + \sum_{i=1}^{k} \beta_i \frac{\log n}{n^i} + g(n), \tag{B.17}$$

is estimated as follows: Including $g(n)$, the quantity obtained by differentiation of Eq. (B.16) becomes

$$C_\infty^{(k)}(n) \approx C + \frac{1}{(k!)^2} \frac{d^k}{dn^k}\left(n^k \frac{d^k}{dn^k}(n^k g(n))\right), \tag{B.18}$$

where we have assumed that the effect of approximation of differentiation by difference is renormalized in $g(n)$ in Eq. (B.18). Thus the error becomes

$$\varepsilon = \left|\frac{1}{(k!)^2} \frac{d^k}{dn^k}\left(n^k \frac{d^k}{dn^k}(n^k g(n))\right)\right|. \tag{B.19}$$

If $g(n)$ can be approximated by a power γ/n^x, we have

$$\varepsilon \equiv |C_\infty^{(k)}(n) - C|$$
$$= \left|\frac{\gamma}{(k!)^2}\left[\frac{\Gamma(x)}{\Gamma(x-k)}\right]^2 \frac{1}{n^x}\right| + O\left(\frac{1}{n^{x+1}}\right) \tag{B.20}$$

and

$$\Delta^{k+1}(n^k \Delta^k(n^k C_n)) \approx \frac{d^k}{dn^x}\left(n^k \frac{d^k}{dn^x}(n^k C_n)\right)$$
$$= -\gamma \frac{\Gamma(x)\,\Gamma(x+1)}{\Gamma(x-k)^2} \frac{1}{n^{x+1}}. \tag{B.21}$$

Thus

$$\varepsilon \approx \frac{1}{(k!)^2}\left|\frac{n}{x} \Delta^{k+1}(n^k \Delta^k(n^k C_n))\right|. \tag{B.22}$$

B.3. *"The Logarithmic Syndrome"*

Let us consider the case in which a function with logarithmic terms is approximated by a polynomial. Assuming that the calculated capacity C_n can be expanded as

$$C(n) = C_\infty + \sum_{i=1}^{\infty} \alpha_i \frac{1}{n^i} + \sum_{i=1}^{\infty} \beta_i \frac{\log n}{n^i}. \tag{B.23}$$

Note that $\log n$ cannot be expanded in terms of $(1/n)$. The extrapolated capacity $C_\infty^{(k)}(n)$ given by Eq. (5) which is obtained by assuming a polynomial of kth degree is

$$C_\infty^{(k)}(n) = C_\infty + \sum_{i=k+1}^{\infty} \alpha_i (-1)^k$$
$$\times \frac{(i-1)!}{k!(k-i+1)!} \frac{1}{n^i} + \sum_{i=1}^{k} \beta_i (-1)^{i-1}$$
$$\times \frac{(i-1)!\,(k-i)!}{k!} \frac{1}{n^i} + O\left(\frac{1}{n^{k+1}}\right). \tag{B.24}$$

If logarithmic terms are absent ($\beta_i = 0$), then $C_\infty^{(k)}(n)$ is of the order of $O(1/n^{k+1})$. In this case, convergence with $n \to \infty$ becomes faster with increasing k. If $\beta_m \neq 0$ at some m, on the other hand, $C_\infty^{(k)}(n)$ is of the order of $O(1/n^m)$ when $k \geq m$. Therefore, the gradient of $\log \Delta C_\infty^{(k)}(n)$ stays $-(m+1)$, even when k is increased. In other words, the syndrome that the gradient of $\log \Delta C_\infty^{(k)}(n)$ does not get steeper with increasing k suggests the necessity of logarithmic terms in the extrapolation function.

APPENDIX C. CAPACITY CALCULATION BETWEEN A PRISM AND A CYLINDER

In two-dimensional systems involving two conductors which do not contact each other, the capacity (per unit length) is defined as the charge (per unit length) on one of the conductors divided by its potential, relative to the other.

Since the system in the present study has m-fold symmetry, we first divide this system into m equivalent parts, then we divide each part into $2n$ smaller parts by a common angle $\theta = 2\pi/(2mn)$. Using complex numbers for notation, we have

$$p_{ab;cd} = -\sum_{f=0}^{m-1} \frac{1}{2\pi\varepsilon_0} \log |P_{ab} - (S_{cd} + S_{cd}^*)\, e^{if\alpha}|, \tag{C.1}$$

where P_{ab} and S_{cd} represent the coordinates of field and source points, respectively, and $\alpha = 2\pi/m$. These points are shown in Fig. 7. For field points, we have

$$P_{1b} = \frac{r_1 \sin \alpha e^{ib\theta}}{\sin(b\theta)(1-\cos\alpha) + \cos(b\theta)\sin\alpha}, \tag{C.2}$$

$$P_{2b} = r_2 e^{ib\theta}; \tag{C.3}$$

while, for the source points, we have

$$S_{1d} = \tfrac{1}{2}(P_{1d} + P_{1,d-1}), \qquad \text{(C.4)}$$

$$S_{2d} = r_2 e^{i(d-(1/2))\theta}. \qquad \text{(C.5)}$$

In Eq. (C.1), the effects of all the other source points are accounted for by rotation $e^{if\alpha}$ and complex conjugation S^*_{cd} of S_{cd} $(1 \leqslant d \leqslant n)$.

REFERENCES

1. J. C. Maxwell, *Electrical Research of the Honorable Henry Cavendish* (Cambridge Univ. Press, Cambridge, 1878), p. 426.
2. A. E. Ruehli and P. A. Brennan, *IEEE Trans. Microwave Theory Tech.* **MTT-21**, 76 (1973).
3. T. Takuma and T. Kawamoto, Central Research Institute of Electric Power Industry Report No. 180029, 1980. [Japanese]
4. D. K. Reitan and T. J. Higgins, *J. Appl. Phys.* **22**, 223 (1951).
5. M. Hosoya, E. Goto, and T. Soma, *J. Comput. Phys.* **74**, 94 (1988); M. Hosoya, Ph.D. thesis, University of Tokyo, 1988 (unpublished).